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INEQUALITIES INVOLVING DERIVATIVES OF THE (p, k)-GAMMA FUNCTION

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Abstract. In this paper, some inequalities involving the m-th derivative of the (p, k)-Gamma function are established. Among other analytical techniques, the procedure makes use of the classical Hölder's, Minkowski's and Chebyshev's integral inequalities.

1. Introduction

The (p, k)-analogue of the Gamma function or simply the (p, k)-Gamma function is defined for $p \in \mathbb{N}$, $k > 0$ and $x \in \mathbb{R}^+$ as [5]

$$\Gamma_{p,k}(x) = \int_0^p t^{x-1}\left(1 - \frac{t^k}{pk}\right)^p dt = \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\dots(x+pk)} \tag{1.1}$$

satisfying the basic relations

$$\Gamma_{p,k}(x+k) = \frac{pkx}{x+pk+k}\Gamma_{p,k}(x), \tag{1.2}$$

$$\Gamma_{p,k}(k) = 1. \tag{1.3}$$

Also, the (p, k)-analogue of the classical Beta function is defined as

$$B_{p,k}(x, y) = \frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)}, \quad x > 0, y > 0. \tag{1.4}$$

Then the m-th derivative of $\Gamma_{p,k}(x)$ is given by

$$\Gamma^{(m)}_{p,k}(x) = \int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{x-1}\, dt \tag{1.5}$$

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where $\Gamma^{(0)}_{p,k}(x) = \Gamma_{p,k}(x)$. The function $\Gamma^{(m)}_{p,k}(x)$ satisfies the commutative diagram:

$$\begin{tikzcd}
\Gamma^{(m)}_{p,k}(x) \arrow[r, "p\to\infty"] \arrow[d, "k\to 1"'] & \Gamma^{(m)}_{k}(x) \arrow[d, "k\to 1"] \\
\Gamma^{(m)}_{p}(x) \arrow[r, "p\to\infty"'] & \Gamma^{(m)}(x)
\end{tikzcd}$$

where $\Gamma_p(x)$ and $\Gamma_k(x)$ are respectively the p and k analogues of the classical Gamma function, $\Gamma(x)$.

In this paper, our goal is to establish some inequalities involving the function $\Gamma^{(m)}_{p,k}(x)$ by using the classical Hölder's, Minkowski's and Chebyshev's integral inequalities among other techniques. Throughout the paper, we shall use notations $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$. We present our results in the following section.

2. Results and Discussion

Theorem 2.1. *Let $p \in \mathbb{N}$, $k > 0$, $u > 1$, $\frac{1}{u} + \frac{1}{w} = 1$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}_0$, m, n even and $\frac{m}{u} + \frac{n}{w} \in \mathbb{N}_0$. Then the inequality*

$$\Gamma^{(\frac{m}{u}+\frac{n}{w})}_{p,k}\left(\frac{x}{u} + \frac{y}{w}\right) \leq \left(\Gamma^{(m)}_{p,k}(x)\right)^{\frac{1}{u}} \left(\Gamma^{(n)}_{p,k}(y)\right)^{\frac{1}{w}} \tag{2.1}$$

is satisfied for $x, y > 0$.

Proof. By (1.5) and the Hölders inequality for integrals, we obtain

$$\begin{aligned}
\Gamma^{(\frac{m}{u}+\frac{n}{w})}_{p,k}\left(\frac{x}{u} + \frac{y}{w}\right) &= \int_0^p (\ln t)^{\frac{m}{u}+\frac{n}{w}} \left(1 - \frac{t^k}{pk}\right)^p t^{(\frac{x}{u}+\frac{y}{w})-1}\, dt \\
&= \int_0^p (\ln t)^{\frac{m}{u}} \left(1 - \frac{t^k}{pk}\right)^{\frac{p}{u}} t^{\frac{x-1}{u}} \cdot (\ln t)^{\frac{n}{w}} \left(1 - \frac{t^k}{pk}\right)^{\frac{p}{w}} t^{\frac{y-1}{w}}\, dt \\
&\leq \left(\int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{x-1}\, dt\right)^{\frac{1}{u}} \\
&\quad \times \left(\int_0^p (\ln t)^n \left(1 - \frac{t^k}{pk}\right)^p t^{y-1}\, dt\right)^{\frac{1}{w}} \\
&= \left(\Gamma^{(m)}_{p,k}(x)\right)^{\frac{1}{u}} \left(\Gamma^{(n)}_{p,k}(y)\right)^{\frac{1}{w}}
\end{aligned}$$

which completes the proof. □

Corollary 2.1. *Let $p \in \mathbb{N}$, $k > 0$, $u > 1$, $\frac{1}{u} + \frac{1}{w} = 1$, $m \in \mathbb{N}_0$ and m even. Then the inequality*

$$\Gamma^{(m)}_{p,k}\left(\frac{x}{u} + \frac{y}{w}\right) \leq \left(\Gamma^{(m)}_{p,k}(x)\right)^{\frac{1}{u}} \left(\Gamma^{(m)}_{p,k}(y)\right)^{\frac{1}{w}} \tag{2.2}$$

holds for $x, y > 0$.

Proof. This follows directly from Theorem 2.1 by letting $m = n$. □

Corollary 2.2. *Let $p \in \mathbb{N}$, $k > 0$, $m \in \mathbb{N}_0$ and m even. Then the inequality*

$$\Gamma_{p,k}^{(m)}\left(\frac{x+y}{2}\right) \leq \sqrt{\Gamma_{p,k}^{(m)}(x)\Gamma_{p,k}^{(m)}(y)} \tag{2.3}$$

holds for $x, y > 0$.

Proof. Let $m = n$ and $u = w = 2$ in Theorem 2.1. □

Corollary 2.3. *Let $p \in \mathbb{N}$, $k > 0$, $m \in \mathbb{N}_0$ and m even. Then the inequality*

$$\Gamma_{p,k}^{(m)}(x)\Gamma_{p,k}^{(m+2)}(x) \geq \left(\Gamma_{p,k}^{(m+1)}(x)\right)^2 \tag{2.4}$$

holds for $x > 0$.

Proof. Let $n = m + 2$, $u = w = 2$ and $x = y$ in Theorem 2.1. □

Corollary 2.4. *Let $p \in \mathbb{N}$, $k > 0$, $a \in \mathbb{N}$ and a odd. Then the inequality*

$$\Gamma_{p,k}^{(a-1)}(x)\Gamma_{p,k}^{(a+1)}(x) \geq \left(\Gamma_{p,k}^{(a)}(x)\right)^2 \tag{2.5}$$

holds for $x > 0$.

Proof. Let $u = w = 2$, $x = y$, $m = a - 1$ and $n = a + 1$ in Theorem 2.1. □

Remark 2.1. By letting $p \to \infty$ as $k \to 1$ in Corollary 2.4, we obtain

$$\Gamma^{(a-1)}(x)\Gamma^{(a+1)}(x) - \left(\Gamma^{(a)}(x)\right)^2 \geq 0$$

which agrees with the main result of [2].

Remark 2.2. By letting $p \to \infty$ as $k \to 1$ in Theorem 2.1, we obtain Theorem 3.1 of [1].

Remark 2.3. Let $m = n = 0$ in Theorem 2.1. Then by allowing $p \to \infty$ as $k \to 1$, we obtain Theorem 5 of [3].

Theorem 2.2. *Let $p \in \mathbb{N}$, $k > 0$, $a > 1$, $\frac{1}{a} + \frac{1}{b} = 1$, $m \in \mathbb{N}_0$ and m even. Then the inequality*

$$\Gamma_{p,k}^{(m)}\left(\frac{x}{a} + \frac{y}{b} + s\right) \leq \left(\Gamma_{p,k}^{(m)}(x+s)\right)^{\frac{1}{a}} \left(\Gamma_{p,k}^{(n)}(y+s)\right)^{\frac{1}{b}} \tag{2.6}$$

is satisfied for $x > 0$, $y > 0$ and $s \geq 0$.

Proof. Similarly by the Hölders inequality, we obtain

$$\begin{aligned}
\Gamma_{p,k}^{(m)}\left(\frac{x}{a} + \frac{y}{b} + s\right) &= \int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{(\frac{x}{a} + \frac{y}{b} + s) - 1}\, dt \\
&= \int_0^p (\ln t)^{\frac{m}{a}} \left(1 - \frac{t^k}{pk}\right)^{\frac{p}{a}} t^{\frac{x+s-1}{a}} (\ln t)^{\frac{m}{b}} \left(1 - \frac{t^k}{pk}\right)^{\frac{p}{b}} t^{\frac{y+s-1}{b}}\, dt \\
&\leq \left(\int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{x+s-1}\, dt\right)^{\frac{1}{a}} \\
&\quad \times \left(\int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{y+s-1}\, dt\right)^{\frac{1}{b}} \\
&= \left(\Gamma_{p,k}^{(m)}(x+s)\right)^{\frac{1}{a}} \left(\Gamma_{p,k}^{(m)}(y+s)\right)^{\frac{1}{b}}
\end{aligned}$$

establishing the result. □

Theorem 2.3. *Let $p \in \mathbb{N}$, $k > 0$, $u \geq 1$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}_0$ and m, n even. Then the inequality*

$$\left(\Gamma_{p,k}^{(m)}(x) + \Gamma_{p,k}^{(n)}(y)\right)^{\frac{1}{u}} \leq \left(\Gamma_{p,k}^{(m)}(x)\right)^{\frac{1}{u}} + \left(\Gamma_{p,k}^{(n)}(y)\right)^{\frac{1}{u}} \tag{2.7}$$

is satisfied for $x, y > 0$.

Proof. We utilize the fact that $a^u + b^u \leq (a+b)^u$, for $a, b \geq 0$, $u \geq 1$, in conjunction with the Minkowski's inequality for integrals. The process is as follows.

$$\begin{aligned}
&\left(\Gamma_{p,k}^{(m)}(x) + \Gamma_{p,k}^{(n)}(y)\right)^{\frac{1}{u}} \\
&= \left(\int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{x-1}\, dt + \int_0^p (\ln t)^n \left(1 - \frac{t^k}{pk}\right)^p t^{y-1}\, dt\right)^{\frac{1}{u}} \\
&= \left(\int_0^p \left[\left((\ln t)^{\frac{m}{u}} \left(1 - \frac{t^k}{pk}\right)^{\frac{p}{u}} t^{\frac{x-1}{u}}\right)^u + \left((\ln t)^{\frac{n}{u}} \left(1 - \frac{t^k}{pk}\right)^{\frac{p}{u}} t^{\frac{y-1}{u}}\right)^u\right] dt\right)^{\frac{1}{u}} \\
&\leq \left(\int_0^p \left[\left((\ln t)^{\frac{m}{u}} \left(1 - \frac{t^k}{pk}\right)^{\frac{p}{u}} t^{\frac{x-1}{u}}\right) + \left((\ln t)^{\frac{n}{u}} \left(1 - \frac{t^k}{pk}\right)^{\frac{p}{u}} t^{\frac{y-1}{u}}\right)\right]^u dt\right)^{\frac{1}{u}} \\
&\leq \left(\int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{x-1}\, dt\right)^{\frac{1}{u}} + \left(\int_0^p (\ln t)^n \left(1 - \frac{t^k}{pk}\right)^p t^{y-1}\, dt\right)^{\frac{1}{u}} \\
&= \left(\Gamma_{p,k}^{(m)}(x)\right)^{\frac{1}{u}} + \left(\Gamma_{p,k}^{(n)}(y)\right)^{\frac{1}{u}}
\end{aligned}$$

completing the proof. □

Theorem 2.4. *Let $p \in \mathbb{N}$, $k > 0$, $m \in \mathbb{N}_0$, $a \in \mathbb{N}_0$, $m \geq a$, and m, a even. Then the inequality*

$$\left(\exp \Gamma_{p,k}^{(m)}(x)\right)^2 \leq \exp \Gamma_{p,k}^{(m-a)}(x) \cdot \exp \Gamma_{p,k}^{(m+a)}(x) \tag{2.8}$$

holds for $x > 0$.

Proof. We proceed as follows:

$$\begin{aligned}
&\frac{\Gamma_{p,k}^{(m-a)}(x) + \Gamma_{p,k}^{(m+a)}(x)}{2} - \Gamma_{p,k}^{(m)}(x) \\
&= \frac{1}{2}\int_0^p (\ln t)^{m-a} \left(1 - \frac{t^k}{pk}\right)^p t^{x-1}\, dt + \frac{1}{2}\int_0^p (\ln t)^{m+a} \left(1 - \frac{t^k}{pk}\right)^p t^{x-1}\, dt \\
&\quad - \int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{x-1}\, dt \\
&= \frac{1}{2}\int_0^p \left[\frac{1}{(\ln t)^a} + (\ln t)^a - 2\right] (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{x-1}\, dt \\
&\geq 0.
\end{aligned}$$

That is

$$\Gamma_{p,k}^{(m-a)}(x) + \Gamma_{p,k}^{(m+a)}(x) \geq 2\Gamma_{p,k}^{(m)}(x).$$

Then by taking exponents we obtain the desired result (2.8). □

Remark 2.4. By letting $p \to \infty$ as $k \to 1$ in Theorem 2.4, we obtain Theorem 3.1 of [4].

The following Lemma which is known in the literature as Chebyshev's integral inequality for synchronous (asynchronous) mappings can be found in [3].

Lemma 2.1. *Let $f, g, h : (a, b) \subset \mathbb{R} \to \mathbb{R}$ be such that $h(t) \geq 0$ for $t \in (a, b)$ and h, hfg, hf and hg are integrable on (a, b). If f and g are are synchronous (asynchronous) on (a, b), that is,if*

$$(f(s) - f(t))\,(g(s) - g(t)) \geq (\leq) 0, \quad \textit{for} \quad s, t \in (a, b)$$

then we have the inequality

$$\int_a^b h(t)\,dt \int_a^b h(t)f(t)g(t)\,dt \geq (\leq) \int_a^b h(t)f(t)\,dt \int_a^b h(t)g(t)\,dt. \tag{2.9}$$

Theorem 2.5. *Let $p \in \mathbb{N}$ and $k > 0$. Further, let α, β and λ be real numbers such that $\alpha, \beta > 0$ and $\alpha > \lambda > -\beta$. If*

$$\lambda(\alpha - \beta - \lambda) \geq (\leq) 0 \tag{2.10}$$

then the inequalities

$$\Gamma_{p,k}^{(m)}(\alpha)\Gamma_{p,k}^{(m)}(\beta) \geq (\leq) \Gamma_{p,k}^{(m)}(\alpha - \lambda)\Gamma_{p,k}^{(m)}(\beta + \lambda) \tag{2.11}$$

hold for $m \in \mathbb{N}_0$ and m even.

Proof. Let $f, g, h : (0, p) \to (0, \infty)$ be defined by

$$f(t) = t^{\alpha-\lambda-\beta}, \quad g(t) = t^{\lambda} \quad \text{and} \quad h(t) = (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{\beta-1}.$$

If condition (2.10) is satisfied, then the functions f and g are synchronous (asynchronous) on $(0, p)$ and then by the Chebyshevs inequality (2.9), we obtain

$$\begin{aligned} &\int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{\beta-1}\,dt \int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{\beta-1}.t^{\alpha-\lambda-\beta}.t^{\lambda}\,dt \\ &\quad \geq (\leq) \int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{\beta-1}.t^{\alpha-\lambda-\beta}\,dt \int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{\beta-1}.t^{\lambda}\,dt. \end{aligned}$$

That is,

$$\begin{aligned} &\int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{\beta-1}\,dt \int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{\alpha-1}\,dt \\ &\qquad \geq (\leq) \int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{\alpha-\lambda-1}\,dt \int_0^p (\ln t)^m \left(1 - \frac{t^k}{pk}\right)^p t^{\beta+\lambda-1}\,dt \end{aligned}$$

which yields the result (2.11). □

Remark 2.5. Let $p \to \infty$ as $k \to 1$ in Theorem 2.5. Then we obtain Theorem 2.2 of [1].

Remark 2.6. If $m = 0$ in (2.11), then we obtain

$$\Gamma_{p,k}(\alpha)\Gamma_{p,k}(\beta) \geq (\leq) \Gamma_{p,k}(\alpha - \lambda)\Gamma_{p,k}(\beta + \lambda)$$

which by (1.4) can be written as

$$B_{p,k}(\alpha, \beta) \geq (\leq) B_{p,k}(\alpha - \lambda, \beta + \lambda).$$

Corollary 2.5. *Let α, λ be real numbers such that $\alpha > 0$ and $|\lambda| < \alpha$. Then the inequality*

$$\left[\Gamma_{p,k}^{(m)}(\alpha)\right]^2 \leq \Gamma_{p,k}^{(m)}(\alpha - \lambda)\Gamma_{p,k}^{(m)}(\alpha + \lambda) \tag{2.12}$$

holds for $m \in \mathbb{N}_0$ and m even.

Proof. This follows from Theorem 2.5 by letting $\alpha = \beta$. Notice that condition (2.10) becomes

$$\lambda(\alpha - \beta - \lambda) = -\lambda^2 \leq 0.$$

Then the result (2.12) follows from (2.11). □

Remark 2.7. It is interesting to notice that, by letting $x = \alpha - \lambda$ and $y = \alpha + \lambda$ in Corollary 2.2, we obtain a result which coincides with (2.12).

Corollary 2.6. *Let $\beta > 0$ and $x, y \geq 0$. Then the inequality*

$$\Gamma_{p,k}^{(m)}(\beta)\Gamma_{p,k}^{(m)}(x + y + \beta) \geq \Gamma_{p,k}^{(m)}(x + \beta)\Gamma_{p,k}^{(m)}(y + \beta) \tag{2.13}$$

holds for $m \in \mathbb{N}_0$ and m even.

Proof. Let $\alpha = x + y + \beta$ and $\lambda = y$ in Theorem 2.5. Then the condition (2.10) becomes

$$\lambda(\alpha - \beta - \lambda) = xy \geq 0$$

and the result (2.13) follows from (2.11). □

Remark 2.8. Let $\Omega_{p,k}(x) = \frac{\Gamma_{p,k}^{(m)}(x+\beta)}{\Gamma_{p,k}^{(m)}(\beta)}$ for $x \geq 0$ and $\beta > 0$. Then by (2.13), the function $\Omega_{p,k}(x)$ is supermultiplicative on $[0, \infty)$. That is, for $x, y \geq 0$, we have

$$\Omega_{p,k}(x + y) \geq \Omega_{p,k}(x)\Omega_{p,k}(y).$$

Remark 2.9. Theorem 2.5, Corollaries 2.5 and 2.6 provide the (p, k) generalizations of some results obtained in [1]. Let $p \to \infty$ as $k \to 1$, then the previous results are recovered.

Theorem 2.6. *Let $p \in \mathbb{N}$, $k > 0$, $x > 0$, $y > 0$, $m \in \mathbb{N}_0$ and m even. If*

$$(x - k)(y - k) \geq (\leq) 0,$$

then

$$\Gamma_{p,k}^{(m)}(2k)\Gamma_{p,k}^{(m)}(x + y) \geq (\leq)\Gamma_{p,k}^{(m)}(x + k)\Gamma_{p,k}^{(m)}(y + k). \tag{2.14}$$

Proof. Let $\alpha = x + y$, $\beta = 2k$ and $\lambda = y - k$ in Theorem 2.5. Then the condition (2.10) becomes

$$\lambda(\alpha - \beta - \lambda) = (x - k)(y - k) \geq (\leq) 0$$

and the result (2.14) follows from (2.11). □

Remark 2.10. Let $m = 0$ in (2.14). Then by using the relation (1.2) and (1.3) noting that $\Gamma_{p,k}(2k) = \frac{pk}{p+2}$, we obtain

$$\Gamma_{p,k}(x + y) \geq (\leq)\frac{p + 2}{pk}.\frac{pkx}{x + pk + k}.\frac{pky}{y + pk + k}\Gamma_{p,k}(x)\Gamma_{p,k}(y) \tag{2.15}$$

which can be written as

$$B_{p,k}(x, y) \geq (\leq)\frac{pk}{p + 2}.\frac{x + pk + k}{pkx}.\frac{y + pk + k}{pky}. \tag{2.16}$$

Remark 2.11. By letting $p \to \infty$, $k \to 1$ in (2.15) and (2.16), we obtain Theorem 3 of [3].

Remark 2.12. Corollary 2.1 implies that the function $\Gamma_{p,k}^{(m)}(x)$ is logarithmically convex for m even.

The following Lemma is found in [6].

Lemma 2.2. *Let $f : (0, \infty) \to (0, \infty)$ be a differentiable, logarithmically convex function. Then the function*

$$g(x) = \frac{(f(x))^{\alpha}}{f(\alpha x)}$$

is decreasing if $\alpha \geq 1$, and increasing if $0 < \alpha \leq 1$.

Theorem 2.7. *Let $p \in \mathbb{N}$, $k > 0$, $m \in \mathbb{N}_0$ and m even. Then the function*

$$T(x) = \frac{[\Gamma_{p,k}^{(m)}(k+x)]^{\alpha}}{\Gamma_{p,k}^{(m)}(k+\alpha x)}, \quad x \geq 0$$

is decreasing if $\alpha \geq 1$ and increasing if $0 < \alpha \leq 1$, and the inequalities

$$\frac{[\Gamma_{p,k}^{(m)}(2k)]^{\alpha}}{\Gamma_{p,k}^{(m)}(k+\alpha k)} \leq \frac{[\Gamma_{p,k}^{(m)}(k+x)]^{\alpha}}{\Gamma_{p,k}^{(m)}(k+\alpha x)} \leq [\Gamma_{p,k}^{(m)}(k)]^{\alpha-1}, \quad \alpha \geq 1 \tag{2.17}$$

and

$$\frac{[\Gamma_{p,k}^{(m)}(2k)]^{\alpha}}{\Gamma_{p,k}^{(m)}(k+\alpha k)} \geq \frac{[\Gamma_{p,k}^{(m)}(k+x)]^{\alpha}}{\Gamma_{p,k}^{(m)}(k+\alpha x)} \geq [\Gamma_{p,k}^{(m)}(k)]^{\alpha-1}, \quad 0 < \alpha \leq 1 \tag{2.18}$$

hold for $x \in [0, k]$.

Proof. Let $f(x) = \Gamma_{p,k}^{(m)}(k+x)$ for $x \geq 0$, $p \in \mathbb{N}$ and $k > 0$. Since $\Gamma_{p,k}^{(m)}(x)$ is logarithmically convex for even m, then $f(x)$ is logarithmically convex. Hence by Lemma 2.2, the function $T(x)$ is decreasing for $\alpha \geq 1$. Then for $x \in [0, k]$, we obtain $T(0) \leq T(x) \leq T(k)$ yielding the result (2.17). Also, $T(x)$ is increasing for $0 < \alpha \leq 1$. Then for $x \in [0, k]$, we obtain $T(0) \geq T(x) \geq T(k)$ yielding the result (2.18). □

Remark 2.13. Let $p \to \infty$ in Theorem 2.7. Then we obtain Theorem 3.1 and Corollary 3.2 of [7].

Remark 2.14. Results similar to Theorem 2.7 for the (q, k)-Gamma function can also be found in Theorem 3.4 and Corollary 3.5 of [7].

3. Conclusion

In the study, some inequalities involving the m-th derivative of the (p, k)-Gamma function are established. Among other analytical techniques, the procedure makes use of the classical Hölder's, Minkowski's and Chebyshev's integral inequalities. From the established results, some previous results are recovered as particular cases.

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