Konuralp Journal of Mathematics
Volume 5 No. 1 pp. 123–131 (2017) ©KJM

HERMITE-HADAMARD TYPE INEQUALITIES FOR THE PRODUCT TWO MAPPINGS WHOSE DERIVATIVES ABSOLUTE VALUES ARE s-CONVEX

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Abstract. In this paper, we extend some estimates of the right hand side of a Hermite-Hadamard type inequality for the product two differentiable functions whose derivatives absolute values are s-convex. Some natural applications to special weighted means of real numbers are given. Finally, an error estimate for the Simpson's formula is also addressed.

1. Introduction

Let $f : [c, d] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in [c, d], a < b$. We consider the well-known Hadamard's inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. The Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. This inequality has several applications in nonlinear analysis and the geometry of Banach spaces, see [8]. In the recent years several extensions and generalizations have been considered for classical convexity. We would like to refer the reader to [1, 3, 5, 9, 12] and references therein for more information. A number of papers have been written on this inequality providing some inequalities analogous to Hadamard's inequality given in (1.1) involving two convex functions, see [2, 10, 11].

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$.

For functions $f : [a, b] \to \mathbb{R}$ that are differentiable on (a, b), Dragomir and Agarwal [4] used the formula,

Date: June 29, 2016 and, in revised form, December 16, 2016.
2010 *Mathematics Subject Classification.* 26D10, 26D15.
Key words and phrases. Hermite-Hadamard inequality, differentiable functions, s-convex functions, convex functions.

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt. \tag{1.2}$$

to prove the following results.

Theorem 1.1. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|$ is convex on $[a, b]$ then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{1.3}$$

Theorem 1.2. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$ then the following inequality holds*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \le\frac{b-a}{2(p+1)^{1/p}}\times\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}.\end{aligned} \tag{1.4}$$

In [7], Hudzik and Maligranda considered, among others, a class of functions which is s-convex. This class is defined in the following way: a function $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex if

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in [0,\infty)$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$. It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$. The following example can be found in [7].

Example 1.1. Let $s \in (0, 1)$ and $a, b, c \in \mathbb{R}$. We define function $f : [0,\infty) \to \mathbb{R}$ as

$$f(x)=\begin{cases}\alpha & x=0\\ \beta x^s+\gamma & x>0\end{cases}$$

It can be easily checked that if $\beta \ge 0$ and $0 \le \gamma \le \alpha$, then f is s-convex.

In [6], Dragomir and Fitzpatrick proved the following variant of Hadamards inequality which holds for s-convex functions:

Theorem 1.3. *Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function, where $s \in (0, 1)$ and let $a, b \in [0,\infty)$, $a < b$. If $f \in L^1[a, b]$, then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{s+1}.$$

Motivated by the above results in the present paper, we extend some estimates of the right hand side of a Hermite-Hadamard type inequality for the product two differentiable functions whose derivatives absolute values are s-convex. Also we give their natural applications to special weighted means of real numbers. Finally, *an* error estimate for the Simpson's formula is also addressed.

2. THE MAIN RESULTS

In this section, we investigate some estimates of the right hand side of a Hermite-Hadamard type inequality involving the weighted arithmetic mean of $f(a)$ and $f(b)$.

Theorem 2.1. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) and $\lambda > 0,\ \mu > 0$. If $|f'|$ is s-convex on $[a, b]$ for some $s \in (0, 1]$, then the following inequality holds*

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le (b-a)\left[\left(\frac{2\mu^{s+2}+(\lambda(s+1)-\mu)(\lambda+\mu)^{s+1}}{(s+1)(s+2)(\lambda+\mu)^{s+2}}\right)|f'(a)|+ \left(\frac{2\lambda^{s+2}+(\mu(s+1)-\lambda)(\lambda+\mu)^{s+1}}{(s+1)(s+2)(\lambda+\mu)^{s+2}}\right)|f'(b)|\right]. \tag{2.1}$$

Proof. Similar to equality (1.2), we have the following equality for a differentiable function f.

$$\int_0^1 \big(\mu-(\lambda+\mu)t\big)f'(ta+(1-t)b)dt = \frac{\lambda f(a)+\mu f(b)}{b-a}-\frac{\lambda+\mu}{b-a}\int_0^1 f(ta+(1-t)b)dt. \tag{2.2}$$

Then

$$\begin{aligned}\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_0^1 f(x)dx\right| &= \left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\int_0^1 f(ta+(1-t)b)dx\right| \\ &= \left|\frac{b-a}{\lambda+\mu}\int_0^1 \big(\mu-(\lambda+\mu)t\big)f'(ta+(1-t)b)dt\right| \\ &\le \frac{b-a}{\lambda+\mu}\int_0^1 |\mu-(\lambda+\mu)t|\,|f'(ta+(1-t)b)|\,dt \\ &\le \frac{b-a}{\lambda+\mu}\int_0^1 |\mu-(\lambda+\mu)t|\,\big(t^s|f'(a)|+(1-t)^s|f'(b)|\big)dt.\end{aligned}$$

Since

$$\begin{aligned}\int_0^1 \big|\mu-(\lambda+\mu)t\big|t^s dt &= \frac{2\mu^{s+2}+(\lambda(s+1)-\mu)(\lambda+\mu)^{s+1}}{(s+1)(s+2)(\lambda+\mu)^{s+1}}, \\ \int_0^1 \big|\mu-(\lambda+\mu)t\big|(1-t)^s dt &= \frac{2\lambda^{s+2}+(\mu(s+1)-\lambda)(\lambda+\mu)^{s+1}}{(s+1)(s+2)(\lambda+\mu)^{s+1}},\end{aligned} \tag{2.3}$$

we get the desired inequality in (2.1). □

Theorem 2.2. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) and $\lambda > 0,\ \mu > 0$. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is s-convex on $[a, b]$ for some $s \in (0, 1]$, then the following inequality holds*

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)(\lambda^{p+1}+\mu^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(\lambda+\mu)^{\frac{p+1}{p}}}\times\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{s+1}\right]^{\frac{p-1}{p}}. \tag{2.4}$$

Proof. Utilizing Holder's inequality

$$\int_0^1\left|\big(\mu-(\lambda+\mu)t\big)f'(ta+(1-t)b)\right|dt \leq \left(\int_0^1\left|\big(\mu-(\lambda+\mu)t\big)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}, \tag{2.5}$$

we obtain inequality in (2.4) from (2.2) and (2.5). □

Theorem 2.3. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f,g:[a,b]\to\mathbb{R}$ be two differentiable functions on (a,b) and $\lambda>0,\ \mu>0$. Assume $q,r\in\mathbb{R}$ with $q,r>1$ and $q+r<qr$.*

(a) *If $|f'|^q,|g|^r,|f|^r,|g'|^q$ are s-convex on $[a,b]$ for some $s\in(0,1]$, then the following inequality holds*

$$\left|\frac{\lambda f(a)g(a)+\mu f(b)g(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)g(x)dx\right| \leq \frac{(b-a)(\mu^{p+1}+\lambda^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(\lambda+\mu)^{\frac{p+1}{p}}}\times\left[\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\frac{|g(a)|^r+|g(b)|^r}{s+1}\right)^{\frac{1}{r}} + \left(\frac{|f(a)|^r+|f(b)|^r}{s+1}\right)^{\frac{1}{r}}\left(\frac{|g'(a)|^q+|g'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right], \tag{2.6}$$

(b) *If $|f'|^q,|g|^r,|f|^q,|g'|^r$ are s-convex on $[a,b]$ for some $s\in(0,1]$, then the following inequality holds*

$$\left|\frac{\lambda f(a)g(a)+\mu f(b)g(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)g(x)dx\right| \leq \frac{(b-a)(\lambda^{p+1}+\mu^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(\lambda+\mu)^{\frac{p+1}{p}}}\times\left[\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\frac{|g(a)|^r+|g(b)|^r}{s+1}\right)^{\frac{1}{r}} + \left(\frac{|f(a)|^q+|f(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\frac{|g'(a)|^r+|g'(b)|^r}{s+1}\right)^{\frac{1}{r}}\right], \tag{2.7}$$

where $p=\frac{qr}{qr-(q+r)}$.

Proof. Using (2.2) for two differentiable functions f and g, we have

$$\int_0^1\big(\mu-(\lambda+\mu)t\big)(fg)'(ta+(1-t)b)dt = \frac{\lambda f(a)g(a)+\mu f(b)g(b)}{b-a}-\frac{\lambda+\mu}{b-a}\int_0^1(fg)(ta+(1-t)b)dt. \tag{2.8}$$

Then

$$\begin{aligned}(2.9)\quad &\left|\frac{\lambda f(a)g(a)+\mu f(b)g(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b (fg)(x)dx\right| \\ &\le \frac{b-a}{\lambda+\mu}\int_0^1 |\mu-(\lambda+\mu)t|\,|(fg)'(ta+(1-t)b)|\,dt \\ &\le \frac{b-a}{\lambda+\mu}\int_0^1 |(\lambda-(\lambda+\mu)t)|\Big[|f'(ta+(1-t)b)||g(ta+(1-t)b)| \\ &\qquad + |f(ta+(1-t)b)||g'(ta+(1-t)b)|\Big]dt.\end{aligned}$$

Utilizing generalized Holder's inequality for three functions, inequalities in (2.6) and (2.7) follow from (2.9). □

Remark 2.1. If we put $s=1$ in the above theorems, then we obtain some interesting results as follows:

(i) Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) and $\lambda>0,\ \mu>0$. If $|f'|$ is convex on $[a,b]$, then

$$\begin{aligned}(2.10)\quad &\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le (b-a)\left[\left(\frac{\mu^3+3\mu\lambda^2+2\lambda^3}{6(\lambda+\mu)^3}\right)|f'(a)|+\right. \\ &\qquad \left.\left(\frac{\lambda^3+3\mu^2\lambda+2\mu^3}{6(\lambda+\mu)^3}\right)|f'(b)|\right].\end{aligned}$$

(ii) Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) and $\lambda>0,\ \mu>0$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a,b]$ for some $p>1$, then

$$\begin{aligned}(2.11)\quad &\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \frac{(b-a)(\mu^{p+1}+\lambda^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(\lambda+\mu)^{\frac{p+1}{p}}}\times\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}.\end{aligned}$$

(iii) Let $f,g:[a,b]\to\mathbb{R}$ be two differentiable functions on (a,b). If $|f'|^q,|f|^r$ and $|g|^r,|g'|^q$ are convex on $[a,b]$ for $r,q>1$ with $q+r<qr$, then

$$\begin{aligned}(2.12)\quad &\left|\frac{\lambda f(a)g(a)+\mu f(b)g(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)g(x)dx\right| \\ &\le \frac{(b-a)(\mu^{p+1}+\lambda^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(\lambda+\mu)^{\frac{p+1}{p}}}\times\left[\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\left(\frac{|g(a)|^r+|g(b)|^r}{2}\right)^{\frac{1}{r}}\right. \\ &\qquad \left.+\left(\frac{|f(a)|^r+|f(b)|^r}{2}\right)^{\frac{1}{r}}\left(\frac{|g'(a)|^q+|g'(b)|^q}{2}\right)^{\frac{1}{q}}\right].\end{aligned}$$

where $p=\frac{qr}{qr-(q+r)}$.

By the same strategy used in the proof of Theorems 2.1 and 2.3, we obtain the following theorem:

Theorem 2.4. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f, g : [a, b] \to \mathbb{R}$ are differentiable functions on (a, b) and $\lambda > 0$, $\mu > 0$. If $|f'|, |f|$ and $|g'|, |g|$ are convex on $[a, b]$, then the following inequality holds*

$$\left| \frac{\lambda f(a)g(a) + \mu f(b)g(b)}{\lambda + \mu} - \frac{1}{b-a}\int_a^b f(x)g(x)dx \right| \tag{2.13}$$
$$\leq (b-a)\left[\left(\frac{\mu^4 + 6\mu^2\lambda^2 + 8\mu\lambda^3 + 3\lambda^4}{12(\lambda+\mu)^4} \right) M(a) + \left(\frac{3\mu^4 + 8\mu^3\lambda + 6\mu^2\lambda^2 + \lambda^4)}{12(\lambda+\mu)^4} \right) M(b) + \left(\frac{\mu^4 + 2\mu^3\lambda + 2\mu\lambda^3 + \lambda^4}{12(\lambda+\mu)^4} \right) \Big(N(a,b) + N(b,a) \Big) \right],$$

where $M(a) = |f'(a)g(a)| + |f(a)g'(a)|$ and $N(a, b) = |f'(a)g(b)| + |f(b)g'(a)|$.

Remark 2.2. If we put $\lambda = \mu = 1$ in inequality (2.10), then we obtain inequality (1.3) and if we put $g = 1$ in inequality (2.13), then this inequality becomes (2.1).

In the following, we give some examples of functions f, g such that $|f|, |f'|$ and $|g|, |g'|$ are convex functions but $|(fg)'|$ is not convex. Therefore function $h = fg$ doesn't satisfy the assumptions (i) of Remark 2.1 and in this case we can't use inequality in (2.10) for h. However, inequality in (2.13) will be available that is an estimate of the right hand side of a Hermite-Hadamard type inequality for function h.

Example 2.1.

(a) We consider real value convex functions $f(x) = x^{-2}, g(x) = x^{\frac{7}{2}}$ on $(0, \infty)$. The functions $|f'(x)| = 2x^{-3}$ and $|g'(x)| = \frac{7}{2}x^{\frac{5}{2}}$ are convex on $(0, \infty)$ but $|(fg)'(x)| = \frac{3}{2}x^{\frac{1}{2}}$ is not convex.

(b) Real value convex functions $f(x) = x, g(x) = \frac{1}{3}x^2 - 4$ on $\mathbb{R}$, in this case $|(fg)'(x)| = |x^2 - 4|$.

(c) Real value convex functions $f(x) = x, g(x) = e^{-x}$ on segment $(1, 4)$, in this case $|(fg)'(x)| = |1 - x|e^{-x}$.

3. APPLICATION

In this sections, we present some applications of our main results.

3.1. **Application to special means.**

We consider some weighted means for arbitrary real numbers a, b $(a \neq b)$ with the weight $w_0 = (\lambda, \mu)$, $\lambda, \mu \geq 0$. We take

1. *Weighted arithmetic mean:*

$$A_{w_0}(a, b) = \frac{\lambda a + \mu b}{\lambda + \mu}, \qquad \lambda + \mu \neq 0,\ a, b \in \mathbb{R},$$

2. *Weighted geometric mean:*

$$G_{w_0}(a,b) = (a^{\lambda} b^{\mu})^{\frac{1}{\lambda+\mu}}, \qquad \lambda+\mu \neq 0,\ a,b \geq 0,$$

3. *Weighted harmonic mean:*

$$H_{w_0}(a,b) = \frac{\lambda+\mu}{\frac{\lambda}{a}+\frac{\mu}{b}}, \qquad \lambda+\mu \neq 0 \qquad a,b \neq 0,$$

4. *Weighted power mean:*

$$M_{p,w_0}(a,b) = \left(\frac{\lambda a^p + \mu b^p}{\lambda+\mu}\right)^{\frac{1}{p}}, \qquad \lambda+\mu \neq 0,\ a,b \geq 0,\ p \neq 0,$$

5. *Logarithmic mean:*

$$L(a,b) = \frac{b-a}{ln|b| - ln|a|}, \qquad |a| \neq |b|,\ a,b \neq 0,$$

6. *Generalized logarithmic mean:*

$$L_p(a,b) = \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \qquad p \in \mathbb{R} \setminus \{0,-1\},\ a \neq b.$$

If $\lambda = \mu$ in the weight function $w_0 = (\lambda, \mu)$, we take A, G, H, M_p for arithmetic mean, geometric mean, harmonic mean, power mean, respectively.

Proposition 3.1. *Let $a,b \in \mathbb{R},\ 0 \leq a < b$ and $p \in \mathbb{R}$. Then*

$$|M^p_{p,w_0}(a,b) - L^p_p(a,b)| \leq |p|(b-a)(w_1+w_2) M^{p-1}_{p-1,w}(|a|,|b|),$$

where $w = (w_1,w_2) = \left(\frac{\mu^3+3\mu\lambda^2+2\mu^3}{6(\lambda+\mu)^3}, \frac{\lambda^3+3\mu^2\lambda+2\mu^3}{6(\lambda+\mu)^3}\right)$, if $p \geq 2$ or $p \leq 1$ and $w = (w_1,w_2) = \left(\frac{2\mu^{s+2}+(\lambda(s+1)-\mu)(\lambda+\mu)^{s+1}}{(s+1)(s+2)(\lambda+\mu)^{s+2}}, \frac{2\lambda^{s+2}+(\mu(s+1)-\lambda)(\lambda+\mu)^{s+1}}{(s+1)(s+2)(\lambda+\mu)^{s+2}}\right)$, if $1 < p < 2$.

Proof. The proof follows from (2.1) and (2.10) applied for $f(x) = x^p, x \in [a,b]$. □

Proposition 3.2. *Let $a,b \in \mathbb{R},\ 0 \leq a < b$ and $p,k \in \mathbb{R},\ p \geq 1$.*

(i) *If $k \geq 2$, or $k \leq 1$, then*

$$|M^k_{k,w_0}(a,b) - L^k_k(a,b)| \leq \frac{(b-a)(\lambda^{p+1}+\mu^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(\lambda+\mu)^{\frac{p+1}{p}}} |k| M_{\frac{p}{p-1}}(|a|^{k-1}, |b|^{k-1}).$$

(ii) *If $1 < k < 2$, then*

$$|M^k_{k,w_0}(a,b) - L^k_k(a,b)| \leq \frac{(b-a)(\lambda^{p+1}+\mu^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(\lambda+\mu)^{\frac{p+1}{p}}}\ 2^{\frac{p-1}{p}} k^{\frac{1}{p}} M_{\frac{p}{p-1}}(|a|^{k-1}, |b|^{k-1}).$$

Proof. The proof follows from (2.4) and (2.11) applied for $f(x) = x^k, x \in [a,b]$. □

Proposition 3.3. *Let $a,b \in \mathbb{R},\ a < b$ and $0 \notin [a,b]$. Then, the following inequality holds:*

$$|H^{-1}_{w_0}(a,b) - L^{-1}(a,b)| \leq (b-a)(w_1+w_2) H^{-1}_w(|a|^2, |b|^2),$$

where $w = (w_1,w_2) = \left(\frac{\mu^3+3\mu\lambda^2+2\mu^3}{6(\lambda+\mu)^3}, \frac{\lambda^3+3\mu^2\lambda+2\mu^3}{6(\lambda+\mu)^3}\right)$.

Proof. The proof follows from (2.10) applied for $f(x) = \frac{1}{x}, x \in [a,b]$. □

Proposition 3.4. *Let $a, b \in \mathbb{R},\ a < b$ and $0 \notin [a, b]$. Then, the following inequality holds:*

$$|H_{w_0}^{-1}(a,b) - L^{-1}(a,b)| \leq \frac{(b-a)(\lambda^{p+1}+\mu^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(\lambda+\mu)^{\frac{p+1}{p}}} M_{\frac{p-1}{p}}(|a|^{-2}, |b|^{-2}).$$

Proof. The proof is immediate from (2.11) applied for $f(x) = \frac{1}{x}, x \in [a, b]$. □

Proposition 3.5. *Let $a, b \in \mathbb{R},\ 0 \leq a < b$ and $q, r \in \mathbb{R},\ q \geq 1,\ r \geq 1,$ with $q + r < qr$. Then, the following inequality holds:*

$$|M_{q+r,w_0}^{q+r}(a,b) - L_{q+r+1}^{q+r+1}(a,b)| \leq \frac{(b-a)(\lambda^{p+1}+\mu^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(\lambda+\mu)^{\frac{p+1}{p}}}$$
$$\times \left[rM_q(|a|^{r-1}, |b|^{r-1})M_r(|a|^q, |b|^q) + qM_q(|a|^r, |b|^r)M_r(|a|^{q-1}, |b|^{q-1})\right].$$

Proof. The proof is immediate from (2.12) applied for $f(x) = x^r,\ g(x) = x^q,\ x \in [a, b]$ and $p = \frac{qr}{qr-(q+r)}$. □

3.2. **An application to Simpson's formula.**

Theorem 3.1. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$, then the following inequality holds*

$$\left|\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{4}\left[\frac{2^{s+3}+3^{s+1}(s-1)}{3^{s+2}(s+1)(s+2)}(|f'(a)|+|f'(b)|) + \frac{2(2+3^{s+1}(2s+1))}{3^{s+2}(s+1)(s+2)}\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \tag{3.1}$$

Proof.

$$\left|\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{1}{2}\left|\frac{f(a)+2f(\frac{a+b}{2})}{3} - \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(x)dx\right| + \frac{1}{2}\left|\frac{2f(\frac{a+b}{2})+f(b)}{3} - \frac{2}{b-a}\int_{\frac{a+b}{2}}^b f(x)dx\right|. \tag{3.2}$$

Using the inequality (2.1) for $\lambda = 1,\ \mu = 2$ and for $\lambda = 2,\ \mu = 1$, we get

$$\left|\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{4}\left(\frac{2^{s+3}+3^{s+1}(s-1)}{3^{s+2}(s+1)(s+2)}|f'(a)| + \frac{2+3^{s+1}(2s+1)}{3^{s+2}(s+1)(s+2)}\left|f'\left(\frac{a+b}{2}\right)\right|\right)$$
$$+ \frac{b-a}{4}\left(\frac{2+3^{s+1}(2s+1)}{3^{s+2}(s+1)(s+2)}\left|f'\left(\frac{a+b}{2}\right)\right| + \frac{2^{s+3}+3^{s+1}(s-1)}{3^{s+2}(s+1)(s+2)}|f'(b)|\right). \tag{3.3}$$

Hence (3.1) follows from (3.3). □

Remark 3.1. If we put $s = 1$ in the above theorem, then we have

$$\left|\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{648}\left(16|f'(a)|+58\left|f'\left(\frac{a+b}{2}\right)\right|+16|f'(b)|\right). \tag{3.4}$$

where $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b) and $|f'|$ is convex on $[a, b]$.

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