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OPTIMAL WEIGHTED GEOMETRIC MEAN BOUNDS OF CENTROIDAL AND HARMONIC MEANS FOR CONVEX COMBINATIONS OF LOGARITHMIC AND IDENTRIC MEANS

LADISLAV MATEJÍČKA

Abstract. In this paper, optimal weighted geometric mean bounds of centroidal and harmonic means for convex combination of logarithmic and identric means are proved. We find the greatest value $\gamma(\alpha)$ and the least value $\beta(\alpha)$ for each $\alpha \in (0,1)$ such that the double inequality:

$$C^{\gamma(\alpha)}(a,b)H^{1-\gamma(\alpha)}(a,b) < \alpha L(a,b)+(1-\alpha)I(a,b) < C^{\beta(\alpha)}(a,b)H^{1-\beta(\alpha)}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$. Here, $C(a,b)$, $H(a,b)$, $L(a,b)$, and $I(a,b)$ denote centroidal, harmonic, logarithmic and identric means of two positive numbers a and b, respectively.

1. Introduction

Recently, means have been the subject of intensive research. In particular, many remarkable inequalities for the centroidal, harmonic, logarithmic and identric means can be found in the literature [4],[12],[13].

We recall some definitions.

The centroidal, harmonic, logarithmic, identric, and weighted geometric means of two positive real numbers a, b, $a \neq b$, are defined, respectively, as follows:

$$C(a,b) = \frac{2(a^2+ab+b^2)}{3(a+b)},$$
$$H(a,b) = \frac{2ab}{(a+b)},$$
$$L(a,b) = \frac{a-b}{\log a - \log b},$$

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$$I(a,b)=\frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{(a-b)}},$$
$$G_\alpha(a,b)=a^\alpha b^{1-\alpha} \quad \text{for} \quad 0\le\alpha\le 1.$$

Means have many applications not only in mathematics, but in physics, economics, meteorology, etc. (see for example [5], [8], [9]).

It is well-known that the following inequalities hold:

$$H(a,b)<L(a,b)<I(a,b)<C(a,b) \quad \text{for positive} \quad a\ne b. \tag{1.1}$$

In the paper [4], authors inspired by (1.1), proved the following theorems:

Theorem 1.1.

$$\alpha_1 C(a,b)+(1-\alpha_1)H(a,b)<L(a,b)<\beta_1 C(a,b)+(1-\beta_1)H(a,b) \tag{1.2}$$

holds for all $a,b>0$, *with* $a\ne b$ *if and only if* $\alpha_1\le 0$, $\beta_1\ge 1/2$.

Theorem 1.2.

$$\alpha_2 C(a,b)+(1-\alpha_2)H(a,b)<I(a,b)<\beta_2 C(a,b)+(1-\beta_2)H(a,b) \tag{1.3}$$

holds for all $a,b>0$, *with* $a\ne b$ *if and only if* $\alpha_2\le 3/(2e)=0.551819$, $\beta_2\ge 5/8$.

Similar double inequality was proved by Alzer and Qiu [1]:

$$\alpha A(a,b)+(1-\alpha_2)G(a,b)<I(a,b)<\beta A(a,b)+(1-\beta)G(a,b) \tag{1.4}$$

holds for all $a,b>0$, with $a\ne b$ if and only if $\alpha\le 2/3$, $\beta\ge 2/e=0.73575$.

In the paper [7] the double inequality

$$\lambda C(a,b)+(1-\lambda)H(a,b)<L^\alpha(a,b)I^{1-\alpha}(a,b)<\Delta C(a,b)+(1-\Delta)H(a,b) \tag{1.5}$$

holds for all $a,b>0$ with $a\ne b$, $\alpha\in(0,1>$ if and only if $\lambda(\alpha)\le 0$ and $\Delta(\alpha)\ge(5-\alpha)/8$.

From results of (1.1), it is natural to ask what is the greatest function $\gamma(\alpha)$, and the least function $\beta(\alpha)$, for $0\le\alpha\le 1$ such that the double inequality:

$$C^{\gamma(\alpha)}(a,b)H^{1-\gamma(\alpha)}(a,b)<\alpha L(a,b)+(1-\alpha)I(a,b)<C^{\beta(\alpha)}(a,b)H^{1-\beta(\alpha)}(a,b)$$

holds for all $a,b>0$ with $a\ne b$. The purpose of this paper is to find the optimal functions $\beta(\alpha)$, $\gamma(\alpha)$. For some other details about means, see [1]-[13] and the related references cited there in.

2. Main Results

Lemma 2.1. *The following inequalities are valid:*

$$d(t)=\frac{-2-9t+t^2-t^3+9t^4+2t^5}{1+5t+12t^2+12t^3+5t^4+t^5}-\ln t>0, \tag{2.1}$$

for $0<t<1$.

$$v(t)=\frac{2}{3t}(1-t+t^2-3t^3)-\ln^2(t)>0, \tag{2.2}$$

for $0<t\le 0.5$.

$$l(t)=\frac{2}{3t}(1-t+t^2-3t^3)(1+8t+6t^2+8t^3+t^4)+(1-t^2)t(1+4t+t^2)\ln(t)-$$
$$(1-t^2)(1+t+t^2)<0, \tag{2.3}$$

for $0 < t \leq 0.3$.

(2.4) $$m(t) = (1 + 8t + 6t^2 + 8t^3 + t^4)\ln(t) + (1 - t^2)(1 + 4t + t^2) < 0,$$

for $0 < t < 1$.

(2.5) $$p(t) = \frac{2 + 3t - 6t^2 + t^3}{6t} + \ln(t) > 0,$$

for $0 < t < 1$.

$$q(t) = (-2-5t+t^2)(1+8t+6t^2+8t^3+t^4)\ln(t)+(1-t^2)(1+4t+t^2)(-2-5t+t^2)-$$

(2.6) $$6(1 + t)(1 + t + t^2) < 0,$$

for $0.3 \leq t < 1$.

Proof. If we show that $d'(t) < 0$ for $0 < t < 1$ then (2.1) will be proved because of $d(1) = 0$. Some calculation gives $d'(t) < 0$ is equivalent to

$$-1 - 9t + t^2 + 38t^3 + 8t^4 - 74t^5 + 8t^6 + 38t^7 + t^8 - 9t^9 - t^{10} < 0.$$

It can be rewritten as

$$(1 - t)^4(1 + 13t + 45t^2 + 68t^3 + 45t^4 + 13t^5 + t^6) > 0.$$

So the proof of (2.1) is complete.

To show that (2.2) it suffices to prove $v'(t) < 0$ because of $v(0.5) = 0.0195$. From

$$v'(t) = \frac{2}{3t^2}(-1 + t^2 - 15t^3) - \frac{2\ln(t)}{t}$$

we have $v'(t) < 0$ is equivalent to

$$v_1(t) = \frac{1}{3t}(-1 + t^2 - 15t^3) - \ln(t) < 0.$$

Some calculation gives $v_1'(t) = 0$ only for one positive root $t_1 = 0.2297$ from $(0, 1)$. $v_1(0.5) = -1.0569$, $v_1(0^+) = -\infty$, $v_1(0.2297) = -0.1674$ imply $v'(t) < 0$, so $v(t) > 0$ for $0 < t \leq 0.5$.

(2.3) is equivalent to

$$l_1(t) = 3\ln(t) + \frac{-1 + 11t - 2t^2 + 17t^3 - 47t^4 - 22t^5 - 46t^6 - 6t^7}{t + 4t^2 - 4t^4 - t^5} < 0.$$

Because of $l_1(0.3) = -0.2368$ it suffices to show that $l_1'(t) > 0$.

$l_1'(t) > 0$ is equivalent to

$$l_2(t) = 3(t + 4t^2 - 4t^4 - t^5)^2 + t(11 - 4t + 51t^2 - 188t^3 - 110t^4 - 276t^5 - 42t^6)\times$$
$$(t + 4t^2 - 4t^4 - t^5) + t(-1 + 11t - 2t^2 + 17t^3 - 47t^4 - 22t^5 - 46t^6 - 6t^7)\times$$
$$(1 + 8t - 16t^3 - 5t^4) > 0.$$

Some calculations give $l_2(t) > 0$ is equivalent to

$$l_3(t) = 1 + 11t - 22t^2 + 22t^3 + 90t^4 - 758t^5 + 500t^6 - 1002t^7 + 609t^8 - 565t^9$$
$$-50t^{10} + 12t^{11} > 0.$$

Using $-1002t^7 > -1002(0.3)^7 = -0.2191374$, $-565t^9 > -565(0.3)^9 = -0.011120895$, $-50t^{10} > -50(0.3)^{10} = -0.000295245$ we obtain

$$l_3(t) > l_4(t) = 0.76944646 + 11t - 22t^2 + 22t^3 + 90t^4 - 758t^5 + 500t^6.$$

From

$$l_4''(t) = -44 + 132t + 1080t^2 - 15160t^3 + 15000t^4.$$

and $t^4 < 0.3t^3$ we have

$$l_4''(t) < l_5(t) = -44 + 132t + 1080t^2 - 10660t^3.$$

Roots of $l_5'(t) = 132 + 2160t - 31980t^2 = 0$ are $t_1 = 0.1064$, $t_2 = -0.0388$. From $l_5(0) = -44$, $l_5(0.3) = -195.02$, $l_5(0.1064) = -30.5691$ we have $l_4''(t) < 0$. From $l_4(0) = 0.76944646$, $l_4(0.3) = 1.9350$ we obtain $l_3(t) > 0$ and the proof of (2.3) is complete. $m(t) < 0$ is equivalent to

$$n(t) = \ln(t) + \frac{(1-t^2)(1+4t+t^2)}{1+8t+6t^2+8t^3+t^4} < 0.$$

Because of $n(1) = 0$ it suffices to show that $n'(t) > 0$. $n'(t) > 0$ is equivalent to

$$n_1(t) = 1 + 12t + 64t^2 + 52t^3 + 30t^4 + 16t^5 + 52t^6 + 34t^7 + 9t^8 > 0,$$

which is evident.

Using $\ln(t) = -\sum_{n=1}^{\infty} \frac{(1-t)^n}{n}$ we obtain

$$\ln(t) > 1 - t + \frac{(1-t)^2}{2} + \frac{(1-t)^3}{3t} = -\frac{2+3t-6t^2+t^3}{6t},$$

so the proof of (2.5) is complete.

$q(t) < 0$ is equivalent to

$$q_1(t) = \ln(t) + \frac{8+25t+31t^2-6t^3-20t^4+4t^5}{2+21t+51t^2+38t^3+36t^4-3t^5-t^6} > 0.$$

Because of $q_1(0.3) = 0.0620$ it suffices to show that $q_1'(t) > 0$. $q_1'(t) > 0$ is equivalent to

$$q_2(t) = 4 - 64t - 47t^2 + 722t^3 + 877t^4 + 92t^5 + 1398t^6 + 2860t^7 + 1358t^8 - 226t^9 - 103t^{10} + 10t^{11} + t^{12} > 0.$$

Evidently

$$q_2(t) > q_3(t) = 4 - 64t - 47t^2 + 722t^3 + 877t^4 + 92t^5 > q_4(t) = 4 - 64t - 47t^2 + 993.38t^3.$$

$(q \geq 0.3)$.

From $q_4''(t) = -94 + 5960.2t$ and $q_4''(0.3) = 1694.1$ we have $q_4''(t) > 0$. It implies $q_4'(t) = -64 - 94t + 2980.1t^2 > 0$ because of $q_4'(0.3) = 176$. So the proof of our lemma is complete. □

Theorem 2.1. *The double inequality*

(2.7)

$$C^{\gamma(\alpha)}(a,b)H^{1-\gamma(\alpha)}(a,b) < \alpha L(a,b) + (1-\alpha)I(a,b) < C^{\beta(\alpha)}(a,b)H^{1-\beta(\alpha)}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$, $\alpha \in (0, 1 >$ if and only if $\beta(\alpha) \geq 1$ and $\gamma(\alpha) \leq \frac{5-\alpha}{8}$.

Proof. Suppose $a, b > 0$ with $a > b$, $\alpha \in (0, 1)$, $t = b/a < 1$. Using

$$\frac{C(a,b)}{a} = \frac{2(1+t+t^2)}{3(1+t)}, \quad \frac{H(a,b)}{a} = \frac{2t}{1+t},$$

$$\frac{L(a,b)}{a} = \frac{1-t}{-\ln t}, \quad \frac{I(a,b)}{a} = \frac{1}{et^{\frac{t}{1-t}}}$$

we can write inequality (2.7) in the form

$$\left(\frac{2(1+t+t^2)}{3(1+t)}\right)^{\gamma(\alpha)}\left(\frac{2t}{1+t}\right)^{1-\gamma(\alpha)} < \alpha\left(\frac{1-t}{-\ln t}\right)+(1-\alpha)\left(\frac{1}{et^{\frac{t}{1-t}}}\right) <$$

$$\left(\frac{2(1+t+t^2)}{3(1+t)}\right)^{\beta(\alpha)}\left(\frac{2t}{1+t}\right)^{1-\beta(\alpha)}.$$

Then the previous inequality can be rewriting as

$$\gamma(\alpha)\ln\left(\frac{1+t+t^2}{3t}\right) < \ln\left(\left(\alpha\frac{1-t^2}{-\ln(t)}+(1-\alpha)\left(\frac{1}{et^{\frac{t}{1-t}}}\right)\right)\left(\frac{1+t}{2t}\right)\right) <$$

$$\beta(\alpha)\ln\left(\frac{1+t+t^2}{3t}\right).$$

Denote

$$a(t,\alpha)=\alpha\frac{(1-t^2)}{-2t\ln t}+(1-\alpha)\frac{1+t}{2et^{\frac{1}{1-t}}}, \tag{2.8}$$

$$b(t)=\frac{1+t+t^2}{3t}, \tag{2.9}$$

for $0<t<1$, $0\le\alpha\le 1$.

We show that

$$g(t,\alpha)=\frac{\ln(a(t,\alpha))}{\ln(b(t))}=\frac{\ln\left\{\left(\alpha\frac{(1-t)}{-\ln t}+(1-\alpha)\frac{1}{et^{\frac{t}{1-t}}}\right)\left(\frac{1+t}{2t}\right)\right\}}{\ln\left(\frac{1+t+t^2}{3t}\right)} \tag{2.10}$$

is a decreasing function on $0<t<1$, for each α such that $0<\alpha\le 1$.

It implies $\gamma(\alpha)=\lim\limits_{t\to 0^+} g(t,\alpha)$ and $\beta(\alpha)=\lim\limits_{t\to 1^-} g(t,\alpha)$ for each α such that $0<\alpha\le 1$, and the theorem will be proved. The monotonicity of $g(t,\alpha)$ will be done, if we prove

$$\frac{\partial g(t,\alpha)}{\partial t}=\frac{\ln(b(t))}{a(t,\alpha)}\frac{\partial a(t,\alpha)}{\partial t}-\frac{b'(t)}{b(t)}\ln(a(t,\alpha))<0 \tag{2.11}$$

on $0<t<1$, for each α such that $0<\alpha\le 1$. Simple calculations give:

$$b'(t)=\frac{t^2-1}{3t^2}<0, \tag{2.12}$$

for $0<t<1$ and

(2.13)

$$\frac{\partial a(t,\alpha)}{\partial t}=\frac{\alpha}{2}\left(\frac{(1+t^2)\ln(t)+1-t^2}{t^2\ln^2 t}\right)+\frac{1-\alpha}{2e}\left(\frac{-1+t-t^2+t^3-(1+t)t\ln(t)}{t(1-t)^2t^{\frac{1}{1-t}}}\right)$$

for $0<t<1$, $0\le\alpha\le 1$.

It is evident that $\frac{\partial g(t,\alpha)}{\partial t}<0$ is equivalent to $H(t,\alpha)<0$, where

$$H(t,\alpha)=b(t)\frac{\partial a(t,\alpha)}{\partial t}\ln(b(t))-b'(t)a(t,\alpha)\ln(a(t,\alpha)) \tag{2.14}$$

for $0<t<1$, $0<\alpha\le 1$.

It suffices to show that $H(t,0)<0$, $H(t,1)<0$ because of

$$\frac{\partial^2 H(t,\alpha)}{\partial \alpha^2} = -\frac{b'(t)\frac{\partial a(t,\alpha)}{\partial \alpha}^2}{a(t,\alpha)} > 0. \tag{2.15}$$

First we prove

$$H(t,0)<0, \qquad H(t,1)<0 \tag{2.16}$$

for $0<t<1$. $H(t,0)<0$ is equivalent to

$$G(t) = \frac{(1+t+t^2)(-1+t-t^2+t^3-(1+t)t\ln t)}{(1+t)^2(1-t)^3}\ln\left(\frac{1+t+t^2}{3t}\right)+ \ln\left(\frac{1+t}{2et^{\frac{1}{1-t}}}\right)<0.$$

If we show $G'(t)>0$ then the proof $H(t,0)<0$ will be done because of $G(1)=0$. $G'(t)>0$ is equivalent to

$$\left\{\frac{-1-4t+t^2-t^3+5t^4}{1-t^2} - \frac{(1+4t+6t^2+4t^3)\ln(t)}{1-t^2}+ \frac{(1+t^2-t^3-t^5+(t+2t^2+2t^3+t^4)\ln t)(-1-5t)}{(1-t)^2(1+t)^2}\right\}\ln\left(\frac{1+t+t^2}{3t}\right)>0. \tag{2.17}$$

(2.17) is equivalent to

$$d(t) = \frac{-2-9t+t^2-t^3+9t^4+2t^5}{1+5t+12t^2+12t^3+5t^4+t^5} - \ln t > 0.$$

It follows from Lemma 1.

Now we show $H(t,1)<0$. $H(t,1)<0$ is equivalent to

$$(1+t+t^2)\left[(1+t^2)\ln(t)+1-t^2\right]\ln\left(\frac{1+t+t^2}{3t}\right)- (1-t^2)^2\ln(t)\ln\left(\frac{1-t^2}{-2t\ln(t)}\right)<0. \tag{2.18}$$

Denote

$$r(t) = \frac{(1+t+t^2)\left[(1+t^2)\ln(t)+1-t^2\right]}{(1-t^2)^2\ln(t)}\ln\left(\frac{1+t+t^2}{3t}\right)- \ln\left(\frac{1-t^2}{-2t\ln(t)}\right).$$

$H(t,1)<0$ will be proved if we show $r'(t)<0$ because of $r(1^-)=0$.

Some calculations give $r'(t)<0$ is equivalent to

$$\Big\{\left[(1-t^4)(t+2t^2)+(1-t^2)(2t^2+2t^3+2t^4)+(1+t+t^2)(4t^2+4t^4)\right]\ln^2(t)+ (1-t^2)\left[(1-t^2)(t+2t^2)+(1-t^2)(1+t+t^2)+(1+t+t^2)(3t^2-1)\right]\ln(t) -(1+t+t^2)(1-t^2)^2\Big\}\ln\left(\frac{1+t+t^2}{3t}\right)<0. \tag{2.19}$$

From (2.19) we have that it suffices to show

$$\left[t+8t^2+6t^3+8t^4+t^5\right]\ln^2(t)+(1-t^2)\left(t+4t^2+t^3\right)\ln(t)-$$

$$(1+t+t^2)(1-t^2) < 0. \tag{2.20}$$

(2.19) is following from Lemma 1.

Now we find the functions $\gamma(\alpha)$, $\beta(\alpha)$.

We have

$$\beta(\alpha) = \lim_{t\to 0^+} \frac{\ln(a(t,\alpha))}{\ln(b(t))} = \lim_{t\to 0^+} \frac{\frac{\partial a(t,\alpha)}{\partial t} b(t)}{a(t,\alpha)b'(t)}, \tag{2.21}$$

$$\gamma(\alpha) = \lim_{t\to 1^-} \frac{\ln(a(t,\alpha))}{\ln(b(t))} = \lim_{t\to 1^-} \frac{\frac{\partial a(t,\alpha)}{\partial t} b(t)}{a(t,\alpha)b'(t)}, \tag{2.22}$$

(2.21) can be rewriting as

$$\beta(\alpha) = \lim_{t\to 0^+} \frac{\alpha e(1-t)^2 t^{\frac{1}{1-t}} \left[(1+t^2)\ln(t) + 1 - t^2\right]}{\left(\alpha e(1-t^2)t^{\frac{1}{1-t}} - (1-\alpha)t(1+t)\ln(t)\right)\ln(t)} +$$

$$\frac{(1-\alpha)\left[-1+t-t^2+t^3-(1+t)t\ln(t)\right]t\ln(t)}{\left(\alpha e(1-t^2)t^{\frac{1}{1-t}} - (1-\alpha)t(1+t)\ln(t)\right)}. \tag{2.23}$$

(2.23) can be rewriting as

$$\beta(\alpha) = \lim_{t\to 0^+} \frac{\frac{\alpha e(1-t)^2 t^{\frac{1}{1-t}}(1+t^2)}{t\ln(t)} + \frac{\alpha e(1-t)^2(1-t)^2 t^{\frac{1}{1-t}}}{t\ln^2(t)} + (1-\alpha)(-1+t-t^2+t^3-(1+t)t\ln t)}{\frac{\alpha e(1-t^2)t^{\frac{1}{1-t}}}{t\ln(t)} - (1-\alpha)(1+t)}$$

It implies $\beta(\alpha) = 1$.

Similarly (2.22) can be rewriting as

$$\gamma(\alpha) = \lim_{t\to 1^-} \frac{3}{(1-t)^3(1+t)^2}\left[\frac{\alpha e(1-t)^2 t^{\frac{1}{1-t}}\left[(1+t^2)\ln(t)+1-t^2\right]}{\left(\alpha e(1-t)t^{\frac{1}{1-t}} - (1-\alpha)t\ln(t)\right)t\ln(t)} + \right.$$

$$\left. \frac{(1-\alpha)\left[-1+t-t^2+t^3-(1+t)t\ln(t)\right]\ln(t)}{\left(\alpha e(1-t^2)t^{\frac{1}{1-t}} - (1-\alpha)t\ln(t)\right)}\right] =$$

$$\lim_{t\to 1^-} \frac{3}{4(1-t)^3}\left\{\frac{\frac{\alpha e(1-t)^2 t^{\frac{t}{1-t}}((1+t^2)\ln(t)+1-t^2)}{\ln(t)} - (1-\alpha)(1-t)(1+t^2)\ln t - (1-\alpha)(1+t)t\ln^2 t}{\alpha e(1-t)t^{\frac{1}{1-t}} - (1-\alpha)t\ln(t)}\right\}$$

Using the following equations:

$$1+t^2 = 2 - 2(1-t) + (1-t)^2,$$

$$\alpha e(1-t)t^{\frac{1}{1-t}} - (1-\alpha)t\ln(t) = (1-t)(1+(1-t)f(t,\alpha)),$$

$$\ln^2(t) = (1-t)^2 + (1-t)^3 + \frac{11}{12}(1-t)^4 + \frac{5}{6}(1-t)^5 + s(\alpha)(1-t)^6,$$

$$\ln^3(t) = -(1-t)^3 - \frac{3}{2}(1-t)^4 - \frac{21}{12}(1-t)^5 + h(\alpha)(1-t)^6,$$

$$t^{\frac{t}{1-t}} = \frac{1}{e} + \frac{1}{2e}(1-t) + \frac{7}{24e}(1-t)^2 + ch(\alpha)(1-t)^3,$$

where $f(t,\alpha)$, $s(\alpha)$, $h(\alpha)$ $ch(\alpha)$ are suitable functions we obtain

$$\gamma(\alpha) = \frac{5-\alpha}{8}.$$

The proof is complete. □

Competing interests

The author declares that he has no competing interests.

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Faculty of Industrial Technologies in Púchov, Trenčín University of Alexander Dubček in Trenčín, I. Krasku 491/30, 02001 Púchov, Slovakia

E-mail address: ladislav.matejicka@tnuni.sk