



## SHAPIRO TYPE INEQUALITIES FOR THE WEINSTEIN AND THE WEINSTEIN-GABOR TRANSFORMS

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ABSTRACT. The aim of this paper is to prove new uncertainty principles for the Weinstein and the Weinstein-Gabor transforms associated with the Weinstein operator defined on the half space  $\mathbb{R}_+^d$  by

$$\Delta_W = \sum_{i=1}^d \frac{\partial}{\partial x_i^2} + \frac{2\alpha + 1}{x_d} \frac{\partial}{\partial x_{d-1}}; \quad d \geq 2, \alpha > -1/2.$$

More precisely, we give a Shapiro-type uncertainty inequality for the Weinstein transform that is, for  $s > 0$  and  $\{\phi_n\}_n$  be an orthonormal sequence in  $L_\alpha^2(\mathbb{R}_+^d)$

$$\sum_{n=1}^N (\| |x|^s \phi_n \|_{L_\alpha^2(\mathbb{R}_+^d)}^2 + \| |\xi|^s \mathcal{F}_W(\phi_n) \|_{L_\alpha^2(\mathbb{R}_+^d)}^2) \geq KN^{1+\frac{s}{2\alpha+d+1}},$$

where  $K$  is a constant which depends only on  $d$ ,  $s$  and  $\alpha$ .

Next, we establish an analogous inequality for the Weinstein-Gabor transform

### 1. INTRODUCTION

H.S. Shapiro proved in [15] a number of uncertainty inequalities for orthonormal sequences that are stronger than corresponding inequalities for a single function. Quantitative versions of H.S. Shapiro's results appeared in a recent article by Ph. Jaming and A. Powell, [10], where in particular the following sharp Mean-Dispersion inequality is obtained. Let  $\{e_k\}_{k \geq 0}$  be an orthonormal sequence in  $L^2(\mathbb{R})$  then for all  $N \geq 0$

$$\sum_{k=0}^N (M(e_k)^2 + \Delta^2(e_k) + M(\mathcal{F}(e_k))^2 + \Delta^2(\mathcal{F}(e_k))) \geq \frac{(N+1)(2N+1)}{4\pi}.$$

The equality is attained for the sequence of Hermite function.

Here,  $M(e_k) = \int_{\mathbb{R}} t |e_k|^2 dt$  and  $\Delta^2(e_k) = \left( \int_{\mathbb{R}} (t - M(e_k))^2 |e_k|^2 dt \right)$ , which are

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called the time mean of  $e_k$ , the variance of  $e_k$  respectively and  $\mathcal{F}$  is the Fourier transform defined for  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  by

$$(1.1) \quad \mathcal{F}(f)(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$$

and extended from  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  to  $L^2(\mathbb{R})$  in the usual way.

Next E. Malinnikova in [11] gives the following Shapiro type inequality which is a generalization of the Mean-Dispersion principle :

let  $s > 0$  and  $\{\phi_n\}_n$  be an orthonormal sequence in  $L^2(\mathbb{R}^d)$ , then

$$\sum_{n=1}^N (\tau_s^s(\phi_n) + \tau_s^s(\mathcal{F}(\phi_n))) \geq CN^{1+s/2d},$$

where C depends only on  $d$  and  $s$ , here  $\tau_s^s(\phi_n) = \int_{\mathbb{R}^d} |x|^s |\phi_n|^2 dx$ .

The purpose of this paper is to extend these type inequalities to the Weintsein and Weintsein-Gabor transforms.

In order to describe our paper, we first need to introduce some notations.

Throughout this paper,  $\alpha$  is a real number,  $\alpha > -1/2$ . We consider the Weinstein operator (also called Laplace-Bessel operator), (see [1, 2]), defined on  $\mathbb{R}^{d-1} \times (0, \infty)$  by

$$\Delta_W = \sum_{i=1}^d \frac{\partial}{\partial x_i^2} + \frac{2\alpha+1}{x_d} \frac{\partial}{\partial x_{d-1}}; \quad d \geq 2, \quad \alpha > -1/2.$$

For  $d > 3$ , the operator  $\Delta_W$  is the Laplace-Beltrami operator on the Riemanian space  $\mathbb{R}^{d-1} \times (0, \infty)$  equipped with the metric [1]

$$ds^2 = x_d^{4\alpha+2/(d-2)} \sum_{i=1}^d dx_i^2.$$

The Weinstein operator has several applications in pure and applied Mathematics especially in Fluid Mechanics (see *e.g.* [5, 14]). For  $1 \leq p \leq \infty$ , we denote by

$L_\alpha^p(\mathbb{R}_+^d)$  the Lebesgue space consisting of measurable functions  $f$  on  $\mathbb{R}_+^d = \mathbb{R}^{d-1} \times \mathbb{R}_+$  equipped with the norm

$$\|f\|_{L_\alpha^p(\mathbb{R}_+^d)} = \left( \int_{\mathbb{R}_+^d} |f(x', x_d)|^p d\mu_\alpha(x', x_d) \right)^{1/p}, \quad 1 \leq p < \infty$$

$$\|f\|_{L_\alpha^\infty(\mathbb{R}_+^d)} = \text{ess sup}_{x \in \mathbb{R}_+^d} |f(x)| < \infty,$$

where for  $x = (x_1, \dots, x_{d-1}, x_d) = (x', x_d)$  and

$$d\mu_\alpha(x) = \frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}} 2^{\alpha+\frac{d-1}{2}} \Gamma(\alpha+1)} dx' dx_d = \frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}} 2^{\alpha+\frac{d-1}{2}} \Gamma(\alpha+1)} dx_1 \dots dx_d.$$

For  $f \in L_\alpha^1(\mathbb{R}_+^d)$ , the Weinstein (or Laplace-Bessel) transform is defined by

$$\mathcal{F}_W(f)(\xi', \xi_d) = \int_{\mathbb{R}_+^d} f(x', x_d) e^{-i\langle x', \xi' \rangle} j_\alpha(x_d \xi_d) d\mu_\alpha(x', x_d),$$

where  $j_\alpha$  is the spherical Bessel function :

$$j_\alpha(z) = \Gamma(\alpha + 1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\alpha + k + 1)} \left(\frac{z}{2}\right)^{2k}, z \in \mathbb{C}.$$

We recall the generalized translation operator  $\mathcal{T}_x$ ,  $x \in \mathbb{R}_+^d$  associated with the Weinstein operator  $\Delta_W$  is defined for a continuous function  $f$  on  $\mathbb{R}_+^d$ , even with respect to the last variable by

$$\mathcal{T}_x f(y) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_0^\pi f\left(x' + y'; \sqrt{x_d^2 + y_d^2 + 2x_d^2 y_d^2 \cos \theta}\right) (\sin \theta)^{2\alpha} d\theta, \quad y \in \mathbb{R}_+^d,$$

where  $x' + y' = (x_1 + y_1, \dots, x_{d-1} + y_{d-1})$ .

Also, we denote by  $L_{\omega_\alpha}^p$ ,  $1 \leq p \leq \infty$  the space of measurable functions  $f$  on  $\mathbb{R}_+^d \times \mathbb{R}_+^d$  with respect to the measure  $d\omega_\alpha(x, y) = d\mu_\alpha(x) d\mu_\alpha(y)$  such that

$$\|F\|_{L_{\omega_\alpha}^p(\mathbb{R}_+^d \times \mathbb{R}_+^d)} = \left( \int_{\mathbb{R}_+^d \times \mathbb{R}_+^d} |F(x, y)|^p d\omega_\alpha(x, y) \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty.$$

$$\|F\|_{L_{\omega_\alpha}^\infty(\mathbb{R}_+^d \times \mathbb{R}_+^d)} = \text{ess sup}_{x, y \in \mathbb{R}_+^d} |F(x, y)| < \infty.$$

For any function  $g \in L_\alpha^2(\mathbb{R}_+^d)$  and any  $y \in \mathbb{R}_+^d$ , we define the modulation of  $g$  by  $y$  as

$$\mathcal{M}_y g := g_y = \mathcal{F}_W \left( \sqrt{\mathcal{T}_y |g|^2} \right),$$

Due to Plancherel theorem and the invariance of the measure  $\mu_\alpha$  under the generalized translation  $\mathcal{T}_x$ , we have for all  $g \in L_\alpha^2(\mathbb{R}_+^d)$

$$\|g_y\|_{L_\alpha^2(\mathbb{R}_+^d)} = \|g\|_{L_\alpha^2(\mathbb{R}_+^d)}.$$

The Weinstein–Gabor transform is defined as follows :

Let  $g$  be in  $L_\alpha^2(\mathbb{R}_+^d)$ , for a function  $f \in L_\alpha^2(\mathbb{R}_+^d)$  we define its Weinstein–Gabor transform by

$$(1.2) \quad \mathcal{G}_g f(x, y) = \int_{\mathbb{R}_+^d} f(s) \overline{\mathcal{T}_{-x} g_y(s)} d\mu_\alpha(s) = f *_W \mathcal{F}_W^{-1} \left( \sqrt{\mathcal{T}_y |g|^2} \right)(x).$$

Here  $*_W$  denotes the convolution product associated with the Weinstein operator given by

$$f *_W g(x) = \int_{\mathbb{R}_+^d} f(y) \mathcal{T}_{-x}(g)(y) d\mu_\alpha(y).$$

The Weinstein Gabor transform satisfies the following properties :

(1) For any  $f, g$  in  $L_\alpha^2(\mathbb{R}_+^d)$ ,

$$(1.3) \quad \|\mathcal{G}_g f\|_{L_{\omega_\alpha}^\infty(\mathbb{R}_+^d \times \mathbb{R}_+^d)} \leq \|f\|_{L_\alpha^2(\mathbb{R}_+^d)} \|g\|_{L_\alpha^2(\mathbb{R}_+^d)}.$$

(2) For any  $f, g$  in  $L_\alpha^2(\mathbb{R}_+^d)$ , we have the following Plancherel-type formula

$$(1.4) \quad \|\mathcal{G}_g f\|_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)} = \|f\|_{L_\alpha^2(\mathbb{R}_+^d)} \|g\|_{L_\alpha^2(\mathbb{R}_+^d)}.$$

For more details see [12].

As we have mentioned above, we will here concentrate on Shapiro-type uncertainty inequality for the Weinstein transform and for the Weinstein-Gabor transform. Our first result is the following inequality :

For  $s > 0$  and  $\phi = \{\phi_n\}_n^\infty$  be an orthonormal sequence in  $L_\alpha^2(\mathbb{R}_+^d)$ , we have

$$\sum_{n=1}^N (\| |x|^s \phi_n \|_{L_\alpha^2(\mathbb{R}_+^d)}^2 + \| |\xi|^s \mathcal{F}_W(\phi_n) \|_{L_\alpha^2(\mathbb{R}_+^d)}^2) \geq K N^{1 + \frac{s}{2\alpha+d+1}},$$

where  $K$  depends only on  $d$ ,  $s$  and  $\alpha$ .

Next, we establish an analogous of the previous inequality for the Weinstein Gabor transform as follows

$$\sum_{n=1}^N (\| |x|^s \mathcal{G}_g \phi_n \|_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)}^2 + \| |\xi|^s \mathcal{G}_g(\phi_n) \|_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)}^2) \geq K' N^{1 + \frac{s}{2\alpha+d+1}},$$

where  $K'$  depends only on  $d$ ,  $s$  and  $\alpha$ .

## 2. SHAPIRO TYPE UNCERTAINTY INEQUALITY FOR THE WEINSTEIN TRANSFORM

In this section we shall prove the above mentioned of Shapiro's inequality for the Weinstein transform. Our proof is inspired from the results of Malinnikova [11].

As consequence of Heisenberg-type inequality for the Weinstein transform see [3, Theorem 3.4], we have the following lemma.

**Lemma 2.1.** *Let  $s > 0$  and let  $\phi = \{\phi_n\}_n$  be an orthonormal sequence in  $L_\alpha^2(\mathbb{R}_+^d)$ . Then there exists  $j_\phi \in \mathbb{Z}$  such that*

$$\max \left( \| |x|^s \phi_n \|_{L_\alpha^2(\mathbb{R}_+^d)}; \| |\xi|^s \mathcal{F}_W(\phi_n) \|_{L_\alpha^2(\mathbb{R}_+^d)} \right) \geq 2^{s(j_\phi - 1)}.$$

**Theorem 2.1.** *Let  $s > 0$  and let  $\phi = \{\phi_n\}_n$  be an orthonormal sequence in  $L_\alpha^2(\mathbb{R}_+^d)$ . Then for all  $N \geq 1$ ,*

$$(2.1) \quad \sum_{n=1}^N (\| |x|^s \phi_n \|_{L_\alpha^2(\mathbb{R}_+^d)}^2 + \| |\xi|^s \mathcal{F}_W(\phi_n) \|_{L_\alpha^2(\mathbb{R}_+^d)}^2) \geq \frac{(4^{2\alpha+d+1} - 1)^{\frac{s}{2\alpha+d+1}} (\Gamma(\alpha + \frac{d+3}{2}))^{\frac{2s}{2\alpha+d+1}}}{2^{\frac{(2\alpha+d)(5+3s)+6s+5}{2\alpha+d+1}}} N^{1 + \frac{s}{2\alpha+d+1}}.$$

*Proof.* Let  $j$  be an integer and  $P_j = \left\{ n : \max \left( \| |x|^s \phi_n \|_{L_\alpha^2(\mathbb{R}_+^d)}^{1/s}; \| |\xi|^s \mathcal{F}_W(\phi_n) \|_{L_\alpha^2(\mathbb{R}_+^d)}^{1/s} \right) \in [2^{j-1}, 2^j] \right\}$ .

First, from Lemma 2.1 there exists  $j_\phi$  such that  $P_j$  is empty for all  $j < j_\phi$ . Let  $j \geq j_\phi$ , then for all  $n \in P_j$ , we have

$$\| |x|^s \phi_n \|_{L_\alpha^2(\mathbb{R}_+^d)} \leq 2^{js}, \quad \| |\xi|^s \mathcal{F}_W(\phi_n) \|_{L_\alpha^2(\mathbb{R}_+^d)} \leq 2^{js}.$$

Let  $N_j$  be the number of element in  $P_j$ , then by [3, Lemma 4.3],

$$N_j \leq 2^{\frac{(2\alpha+d)(4-s)+4+s}{s}} \left( \Gamma \left( \alpha + \frac{d+3}{2} \right) \right)^{-2} 4^{j(2\alpha+d+1)}.$$

The number of elements in  $\bigcup_{j=j_\phi}^k P_j$  is less than  $c_{s,\alpha,d} 4^{k(2\alpha+d+1)}$  where

$$c_{s,\alpha,d} = \frac{2^{\frac{(2\alpha+d)(4+s)+4+3s}{s}}}{4^{2\alpha+d+1} - 1} \left( \Gamma \left( \alpha + \frac{d+3}{2} \right) \right)^{-2}$$

is a constant that does not depend on  $k$ .

Now, if  $N \geq 2c_{s,\alpha,d} 4^{j_\phi(2\alpha+d+1)}$ , then there exists  $k_\phi > j_\phi$  such that

$$2c_{s,\alpha,d} 4^{(k_\phi-1)(2\alpha+d+1)} \leq N < 2c_{s,\alpha,d} 4^{k_\phi(2\alpha+d+1)}.$$

Therefore at least half elements of the set  $\{1, \dots, N\}$  does not belong to  $\bigcup_{j=j_\phi}^{k-1} P_j$  and then

$$\begin{aligned} \sum_{n=1}^N \left( \| |x|^s \phi_n \|_{L_\alpha^2(\mathbb{R}_+^d)}^2 + \| |\xi|^s \mathcal{F}_W(\phi_n) \|_{L_\alpha^2(\mathbb{R}_+^d)}^2 \right) &\geq \sum_{n=1}^N \max \left( \| |x|^s \phi_n \|_{L_\alpha^2(\mathbb{R}_+^d)}^2, \| |\xi|^s \mathcal{F}_W(\phi_n) \|_{L_\alpha^2(\mathbb{R}_+^d)}^2 \right) \\ &\geq \frac{N}{2} 4^{s(k_\phi-1)} \\ &\geq \frac{N}{2^{2s+1}} \left( \frac{N}{2c_{s,\alpha,d}} \right)^{\frac{s}{2\alpha+d+1}} \\ &= \frac{1}{2^{2s+1} \times (2c_{s,\alpha,d})^{\frac{s}{2\alpha+d+1}}} N^{1+\frac{s}{2\alpha+d+1}}. \end{aligned}$$

If  $N < 2c_{s,\alpha,d} 4^{j_\phi(2\alpha+d+1)}$ , then from Lemma 2.1

$$\begin{aligned} \sum_{n=1}^N \left( \| |x|^s \phi_n \|_{L_\alpha^2(\mathbb{R}_+^d)}^2 + \| |\xi|^s \mathcal{F}_W(\phi_n) \|_{L_\alpha^2(\mathbb{R}_+^d)}^2 \right) &\geq \sum_{n=1}^N \max \left( \| |x|^s \phi_n \|_{L_\alpha^2(\mathbb{R}_+^d)}^2, \| |\xi|^s \mathcal{F}_W(\phi_n) \|_{L_\alpha^2(\mathbb{R}_+^d)}^2 \right) \\ &\geq N 2^{2s(j_\phi-1)} \\ &\geq \frac{N}{2^{2s}} \left( \frac{N}{2c_{s,\alpha,d}} \right)^{\frac{s}{2\alpha+d+1}} \\ &= \frac{1}{2^{2s} \times (2c_{s,\alpha,d})^{\frac{s}{2\alpha+d+1}}} N^{1+\frac{s}{2\alpha+d+1}}. \end{aligned}$$

Then, we get the desired result.  $\square$

### 3. SHAPIRO TYPE UNCERTAINTY INEQUALITY FOR THE WEINSTEIN-GABOR TRANSFORM

In this section we will show the Shapiro-type uncertainty inequality for the Weinstein-Gabor transform. In order to prove this, we introduce a pair of orthogonal projections on  $L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)$ .

Let  $g \in L_\alpha^2(\mathbb{R}_+^d)$  be a nonzero function such that  $\|g\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 = 1$ . We define the orthogonal projection  $\mathcal{P}_g$  by

$$\mathcal{P}_g : L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d) \rightarrow L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d),$$

$$(3.1) \quad \mathcal{P}_g F(x, y) = \int_{\mathbb{R}_+^d \times \mathbb{R}_+^d} F(\xi, \nu) W_g(\xi, \nu; x, y) d\omega_\alpha(\xi, \nu),$$

where

$$W_g(\xi, \nu; x, y) = \frac{1}{\|g\|_{L_\alpha^2(\mathbb{R}_+^d)}^2} \mathcal{T}_x \check{g}_y * \mathcal{W} \check{g}_\nu(-\xi) = \frac{1}{\|g\|_{L_\alpha^2(\mathbb{R}_+^d)}^2} \overline{\mathcal{G}_g(g_{x,y})(\xi, \nu)}.$$

Let  $U \subset \mathbb{R}_+^d \times \mathbb{R}_+^d$  with  $\omega_\alpha(U) < \infty$  and

$$\mathcal{P}_U : L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d) \rightarrow L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d),$$

the orthogonal projection from  $L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)$  onto the subspace of function supported in the subset  $U$  i.e.

$$\mathcal{P}_U F = \chi_U F, \quad F \in L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d).$$

A straightforward computation, as in [12] shows that  $\mathcal{P}_U \mathcal{P}_g$  is a Hilbert-Schmidt operator with norm satisfying

$$(3.2) \quad \|\mathcal{P}_U \mathcal{P}_g\|_{HS}^2 \leq \omega_\alpha(U).$$

We shall use the following theorem which is similar to Theorem 2 in [11].

**Theorem 3.1.** *Let  $\{\phi_n\}_{n=1}^N$  be an orthonormal system of  $L_\alpha^2(\mathbb{R}_+^d)$  and  $U$  be measurable subset of  $\mathbb{R}_+^d \times \mathbb{R}_+^d$  such that  $0 < \omega_\alpha(U) < \infty$ . Assume that*

$$\|\mathcal{G}_g \phi_n\|_{L_{\omega_\alpha}^2(U)}^2 = 1 - a_n^2.$$

Then

$$\sum_{n=1}^N (1 - a_n) \leq \omega_\alpha(U).$$

*Proof.* A standard estimate of the trace of the Hilbert-Schmidt operator  $\mathcal{P}_U \mathcal{P}_g$  (see e.g. [9, Theorem 5.6, p. 63]) gives,

$$\text{tr}(\mathcal{P}_U \mathcal{P}_g) = \|\mathcal{P}_U \mathcal{P}_g\|_{HS}^2.$$

Then, for all  $N \geq 1$ ,

$$\sum_{n=1}^N \langle \mathcal{P}_U \mathcal{G}_g \phi_n, \mathcal{G}_g \phi_n \rangle_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)} = \sum_{n=1}^N \langle \mathcal{P}_U \mathcal{P}_g \mathcal{G}_g \phi_n, \mathcal{G}_g \phi_n \rangle_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)} \leq \text{tr}(\mathcal{P}_U \mathcal{P}_g) \leq \omega_\alpha(U).$$

On the other hand, by Cauchy-Schwartz inequality,

$$\langle \mathcal{P}_U \mathcal{G}_g \phi_n, \mathcal{G}_g \phi_n \rangle_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)} = 1 - \langle \mathcal{P}_U^c \mathcal{G}_g \phi_n, \mathcal{G}_g \phi_n \rangle_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)} \geq 1 - a_n.$$

Thus, for all  $N \geq 1$ ,

$$(3.3) \quad \sum_{n=1}^N (1 - a_n) \leq \omega_\alpha(U).$$

□

**Corollary 3.1.** *Let  $\epsilon \in (0, 1)$  and let  $\{\phi_n\}_{n=1}^N$  be an orthonormal system in  $L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)$  such that  $\mathcal{G}_g \phi_n$  is  $\epsilon$ -concentrated on the ball  $B_{r_o} = \{(x, y) \in \mathbb{R}_+^d \times \mathbb{R}_+^d, |(x, y)| < r_o\}$ , i.e.*

$$\int_{|(x, y)| < r_o} |\mathcal{G}_g \phi_n|^2 \geq 1 - \epsilon^2.$$

Then

$$N \leq \frac{r_o^{2(2\alpha+d+1)}}{(1-\epsilon)2^{2\alpha+d+1}\Gamma(2\alpha+d+2)}.$$

Consequently we obtain the following lemma.

**Lemma 3.1.** *Let  $s, J > 0$  and  $\{\phi_n\}_{n=1}^N$  be an orthonormal system in  $L_\alpha^2(\mathbb{R}_+^d)$  that satisfies*

$$\| |(x, y)|^s \mathcal{G}_g \phi_n \|_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)}^2 \leq J^{2s}.$$

Then

$$(3.4) \quad N \leq \frac{2^{\frac{(2\alpha+d)(4-s)+4+s}{s}}}{3\Gamma(2\alpha+d+2)} (J)^{2(2\alpha+d+1)}.$$

*Proof.* Since

$$\begin{aligned} \int_{|(x,y)| \geq 4^{1/s} J} |\mathcal{G}_g \phi_n(x, y)|^2 d\mu_\alpha(x) &= \int_{|(x,y)| \geq 4^{1/s} J} |(x, y)|^{-2s} |(x, y)|^{2s} |\mathcal{G}_g \phi_n(x, y)|^2 d\mu_\alpha(x) \\ &\leq \frac{1}{4^2 J^{2s}} \| |(x, y)|^s \mathcal{G}_g \phi_n \|_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)}^2 \leq \frac{1}{16}, \end{aligned}$$

then  $\phi_n$  is  $\frac{1}{4}$ -concentrated on the ball  $B_{\frac{1}{4} J}$ . Applying Corollary 3.1, we obtain the desired result.  $\square$

**Lemma 3.2.** *Let  $s > 0$  and let  $\phi = \{\phi_n\}_n$  be an orthonormal sequence. Then there exists  $i_0 \in \mathbb{Z}$  such that*

$$\| |(x, \xi)|^s \mathcal{G}_g(\phi_n) \|_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)} \geq 2^{s(i_0-1)}.$$

It is a consequence of [3, Inequality (5.50)].

**Theorem 3.2.** *Let  $s > 0$  and let  $\phi = \{\phi_n\}_n$  be an orthonormal sequence in  $L_\alpha^2(\mathbb{R}_+^d)$ . Then for all  $N \geq 1$ ,*

$$(3.5) \quad \sum_{n=1}^N \| |(x, \xi)|^s \mathcal{G}_g(\phi_n) \|_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)}^2 \geq \frac{(3(4^{2\alpha+d+1} - 1))^{\frac{s}{2\alpha+d+1}} (\Gamma(2\alpha+d+2))^{\frac{s}{2\alpha+d+1}}}{2^{\frac{(2\alpha+d)(5+3s)+5+6s}{2\alpha+d+1}}} N^{1+\frac{s}{2\alpha+d+1}}.$$

*Proof.* For  $i \in \mathbb{Z}$ , let  $A_i = \{\phi_n : \| |(x, \xi)|^s \mathcal{G}_g \phi_n \|_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)} \in [2^{i-1}, 2^i]\}$ . First, let  $i_0$  as in Lemma 3.2, we can see that  $A_i$  is empty for all  $i < i_0$ .

Let  $i \geq i_0$ . Then for all  $\phi_n \in A_i$ , we have

$$\| |(x, \xi)|^s \mathcal{G}_g \phi_n \|_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)} \leq 2^{is}.$$

Let  $N_i$  be the number of elements in  $A_i$ , then by Lemma 3.1,

$$N_i \leq \frac{2^{\frac{(2\alpha+d)(4-s)+4+s}{s}}}{3\Gamma(2\alpha+d+2)} 4^{i(2\alpha+d+1)}.$$

The number of elements in  $\bigcup_{i=i_0}^k A_i$  is less than  $R_{s,\alpha,d} 4^{k(2\alpha+d+1)}$  where

$$R_{s,\alpha,d} = \frac{2^{\frac{(2\alpha+d)(4+s)+4+3s}{s}}}{3(4^{(2\alpha+d+1)} - 1)\Gamma(2\alpha+d+2)}$$

is a constant that does not depend on  $k$ .

Now, if  $N \geq 2R_{s,\alpha,d} 4^{i_0(2\alpha+d+1)}$ , then there exists  $k_0 > i_0$  such that

$$2R_{s,\alpha,d} 4^{(k_0-1)(2\alpha+d+1)} \leq N < 2R_{s,\alpha,d} 4^{k_0(2\alpha+d+1)}.$$

Therefore at least half elements of the set  $\{1, \dots, N\}$  does not belong to  $\bigcup_{i=i_0}^{k-1} A_i$ , then

$$\begin{aligned}
\sum_{n=1}^N \|||(x, \xi)|^s \mathcal{G}_g \phi_n\|_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)}^2 &\geq \frac{N}{2} 2^{2s(k_0-1)} \\
&\geq \frac{N}{2^{2s+1}} \left( \frac{N}{2R_{s,\alpha,d}} \right)^{\frac{s}{2\alpha+d+1}} \\
&= \frac{1}{2^{2s+1} \times (2R_{s,\alpha,d})^{\frac{s}{2\alpha+d+1}}} N^{1+\frac{s}{2\alpha+d+1}}.
\end{aligned}$$

If  $N < 2R_{s,\alpha,d} 4^{i_0(2\alpha+d+1)}$ , then from Lemma 3.2

$$\begin{aligned}
\sum_{n=1}^N \|||(x, \xi)|^s \mathcal{G}_g \phi_n\|_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)}^2 &\geq N 2^{2s(i_0-1)} \\
&\geq \frac{N}{2^{2s}} \left( \frac{N}{2R_{s,\alpha,d}} \right)^{\frac{s}{2\alpha+d+1}} \\
&= \frac{1}{2^{2s} \times (2R_{s,\alpha,d})^{\frac{s}{2\alpha+d+1}}} N^{1+\frac{s}{2\alpha+d+1}}.
\end{aligned}$$

Then, we get the desired result.  $\square$

**Corollary 3.2.** *Let  $s > 0$  and let  $\phi = \{\phi_n\}_n$  be an orthonormal sequence in  $L_\alpha^2(\mathbb{R}_+^d)$ . Then for all  $N \geq 1$ ,*

$$\begin{aligned}
&\sum_{n=1}^N (\|||x|^s \mathcal{G}_g \phi_n\|_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)}^2 + \|||\xi|^s \mathcal{G}_g(\phi_n)\|_{L_{\omega_\alpha}^2(\mathbb{R}_+^d \times \mathbb{R}_+^d)}^2) \\
&\geq \frac{(3(4^{2\alpha+d+1} - 1))^{\frac{s}{2\alpha+d+1}} (\Gamma(2\alpha + d + 2))^{\frac{s}{2\alpha+d+1}}}{2^{\frac{(2\alpha+d)(5+5s)+5+8s}{2\alpha+d+1}}} N^{1+\frac{s}{2\alpha+d+1}}.
\end{aligned}$$

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